UDK 532.5

NUMERICAL IMPLEMENTATION OF THE METHOD OF BOUNDARY INTEGRAL EQUATIONS IN BOUNDARY VALUE PROBLEMS OF HYDRODYNAMICS

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INTRODUCTION

By characteristics manifold nonlinearity and fundamental problems of continuum mechanics, a significant development, along with the physical, got a computer experiment. Significant achievements were obtained in numerical analysis and, especially, in the numerical implementation of specific mathematical models of mechanics, gas dynamics in a significant development of computational fluid dynamics was given the physical-chemical processes that allow us to offer our customers develop methodical software determining the aerodynamic characteristics of aircraft operated on transonic flight regimes, create engineering calculation methods based on the typical computer-PC in real time the dynamics of a viscous incompressible fluid, the main attention is paid to a new qualitative research methods initial-boundary value problems, which in turn make it possible to formulate realistic mathematical problems and outline some of the ways their permission, give rise to new formulations of mathematical models and solutions of the motion of a viscous fluid and for small and medium Reynolds numbers that are crucial in matters of livelihood and ecology in the complex aerodynamics of lifting surfaces based on the systematic use of the method of boundary integral equations and numerical implementation options obtained and distributed the total nonlinear aerodynamic characteristics of load-bearing forms, plane and space, and studied the processes that accompany flow separation, formation and stability of vortex structures, which allows for extensive theoretical and experimental study of the problems of livelihood and safety, as well as creating an inexhaustible source of energy in the form aero-and hydroxyl units with high efficiency and a wide range of capacities.

THE IDEOLOGY OF THE METHOD AND THE INTEGRAL REPRESENTATION OF SOLUTIONS

The most reliable and proven mathematical model of a viscous fluid is the initial-boundary value problem for systems of differential equations in partial derivatives of the Navier-Stokes equations. This system was built in 1826 and so far not found a general method for investigating and solving this nonlinear system, and are known only to some, usually obtained by chance, the particular solutions of this system. Particular exact solutions are valuable for the study of viscous liquids, because they can figure out the error results when the assumptions made, or another numerical method to verify. Methodological value of such solutions is also great.

Exact analytical solutions of the complete Navier-Stokes equations exist for only a few very specialized cases. Instead, the equations are frequently simplified by making appropriate approximations about the flow. In modern times, exact solutions of the complete Navier-Stokes equations for many practical problems can be obtained numerically, using various techniques of computational fluid dynamics.

Initial-boundary value problems for systems of differential equations in partial derivatives of the Navier-Stokes equations are important and challenging task in applied mathematics and mechanics and their decision will significantly change the way of the hydro-and aerodynamic calculations, improve the quality of these calculations and increase the reliability of the results that may be and also the real economic value.

The boundary integral approach has obvious advantages over finite difference and finite element methods. This is why nowadays this method is being successfully applied for solving complex engineering problems – on surface and in space, stationary and time-dependent.

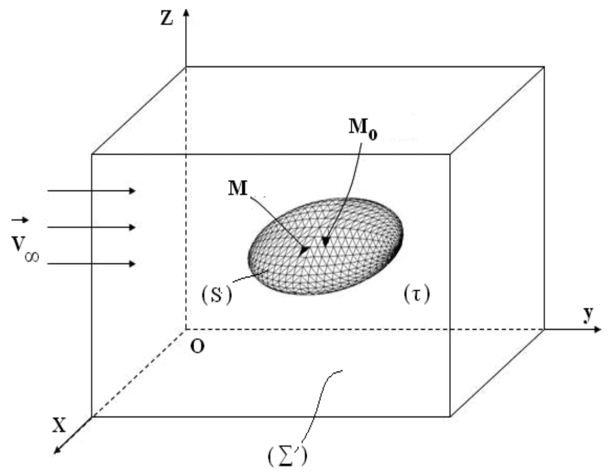

Fig.1 Fixed body in a steady flow of a viscous incompressible fluid inside the control volume.

The stationary problem [1] for the flow of the incompressible viscous fluid around a body is shown in Fig.1. The most effective method for solving a wide spectrum of boundary value problems of continuum mechanics is the method of boundary integral equations [2]. In the absence of internal moments and temperature effects, a mathematical model of the dynamics of an incompressible fluid flow is described using a well-known system of conservation laws:

- mass

$$\left(\nabla,\mathbf{V}\right)=0, \tag{1}$$

where V is velocity of fluid flow;

- momentum

$$\left(\nabla,\left(\mathbf{V}*\mathbf{V}\right)\right)=\left(\nabla,\left(-\mathbf{I}\frac{p}{\rho}+\nu\left(\nabla\mathbf{V}+\nabla^{*}\mathbf{V}\right)\right)\right), \tag{2}$$

where $\nabla^{*}\mathbf{V}$ is the dual tensor $\nabla\mathbf{V}$, and is density of the medium, p is pressure, is kinematical viscosity and I is an identity tensor.

In addition, it is advisable to investigate the conservation of vortices $\Omega=\left[\nabla,\mathbf{V}\right]$

$$\left[\nabla,\left[\mathbf{\Omega},\mathbf{V}\right]\right]+\nu\left[\nabla,\left[\nabla,\mathbf{\Omega}\right]\right]=0 \tag{3}$$

as a critical kinematics haracteristics of the interaction of viscous flow with a streamlined body.

Solution of the system of differential equations of the conservation laws (1 - 3) is subject to the natural boundary condition

$$\mathbf{V}\big|_{S+\Sigma}=\mathbf{U}^{S+\Sigma}\left(s,\tau\right),\ \ p\big|_{\Sigma}=p_{\infty},\ \ \mathbf{\Omega}_n\big|_{S}=0,\ \ \mathbf{\Omega}\big|_{\Sigma}=0, \tag{4}$$

where $\mathbf{V}\big|_{S}$ is the velocity of points on the liquid surface of the body and is a boundary condition that may depend on the surface coordinates in some particular cases.

The main objective of the vector analysis is the determination of the vector field on the set of divergence

$$\left(\nabla,\mathbf{V}\right)=q \tag{5}$$

and vorticity

$$\left[\nabla,\mathbf{V}\right]=\mathbf{\Omega}, \tag{6}$$

where the intensity function of the density of mass flow q - known as the vortices vector Ω - near-contains the definition.

Thus, in accordance with the theorems of vector analysis [3] and with the theory of boundary value problems of mathematical physics [4], solutions must be consistent with the natural boundary conditions of the mathematical model (1 - 4). Therefore, it is necessary to solve the boundary value problem (1 - 4), for example, in R^3.

Then equations (5 - 6) can be constructed using the differential operators of second order:

$$\nabla\left(\nabla,\mathbf{V}\right)=\nabla q,\quad \nabla\left(\nabla,\mathbf{\Omega}\right)=0. \tag{7}$$

It is not difficult to prove [4-5] that the tensor

$$\mathbf{\Gamma}=\mathbf{I}\varphi-\left[\mathbf{I},\mathbf{G}\right], \tag{8}$$

where

$$\varphi\left(\left|\mathbf{x}-\mathbf{y}\right|\right)=\frac{1}{4\pi\left|\mathbf{x}-\mathbf{y}\right|}$$

is the fundamental solution of Laplace's equation in R^3, and the vector $\mathbf{G}\in C^2\left(E\right)$ defined by the conservative condition

$$\left(\nabla,\mathbf{\Gamma}\right)=0\Leftrightarrow\nabla\varphi=\left[\nabla,\mathbf{G}\right] \tag{9}$$

is the fundamental solution of differential operators in (7), i.e.

$$\nabla\left(\nabla,\mathbf{\Gamma}\right)=\mathbf{I}\delta\left(\left|\mathbf{x}-\mathbf{y}\right|\right), \tag{10}$$

where $\delta\left(\left|\mathbf{x}-\mathbf{y}\right|\right)$ is the Dirac delta function, which depends on two points in space. In spherical coordinates ($\theta\in[0;\pi]$; $\omega\in[0;2\pi]$) the vector G used in one of the following:

$$\mathbf{G}_1=-\frac{\omega}{4\pi}\nabla\left(\cos\theta\right);\ \mathbf{G}_2=\frac{\cos\theta}{4\pi}\nabla\omega. \tag{11}$$

Conservative nature of the law of conservation of the momentum (2) allows us to write the generalized D.Generalized Bernoulli integral:

$$\mathbf{V} * \mathbf{V} + \mathbf{I}\frac{p}{\rho} + \nu\left(\nabla^{*}\mathbf{V} - \nabla\mathbf{V}\right) = \nabla^{*}\mathbf{\Psi}, \quad (12)$$

where the choice of the vector potential Ψ, by virtue of conservatism of the left side of expression (2) will be governed by the condition

$$\left(\nabla, \left(\nabla^{*}\mathbf{\Psi}\right)\right) \equiv \nabla\left(\nabla, \mathbf{\Psi}\right) = 0, \quad (13)$$

i.e. vector Ψ belongs to the class of solutions of equations (7).

Similarly, the law of conservation of vortices Ω (see (3)) has an integral representation

$$\left[\mathbf{I}, \left[\mathbf{\Omega}, \mathbf{V}\right]\right] + \nu\left(\nabla^{*}\mathbf{\Omega} - \nabla\mathbf{\Omega}\right) = \nabla^{*}\mathbf{\Phi}, \quad (14)$$

which is a vector potential and it is a solution of equation (13)

$$\left(\nabla, \left(\nabla^{*}\mathbf{\Phi}\right)\right) \equiv \nabla\left(\nabla, \mathbf{\Phi}\right) = 0. \quad (15)$$

For further development of the theory of boundary integral equations prove the following generalized vector-operations of tensor analysis [3, 4]:

$$\left(\nabla, \left(\mathbf{I}\varphi\right)\right) = \nabla\varphi; \quad (16)$$

$$\left(\nabla, \left[\mathbf{I}, \mathbf{a}\right]\right) = \left[\nabla, \mathbf{a}\right]; \quad (17)$$

$$\left[\nabla, \mathbf{I}\varphi\right] = \left[\mathbf{I}, \nabla\varphi\right]; \quad (18)$$

$$\left[\nabla, \left[\mathbf{I}, \mathbf{a}\right]\right] = \nabla^{*}\mathbf{a} - \mathbf{I}\left(\nabla, \mathbf{a}\right); \quad (19)$$

$$\left[\mathbf{I}, \left[\nabla, \mathbf{a}\right]\right] = \nabla^{*}\mathbf{a} - \nabla\mathbf{a}; \quad (20)$$

$$\nabla\left(\nabla, \mathbf{a}\right) = \left(\nabla, \nabla^{*}\mathbf{a}\right), \quad (21)$$

where the vector function a has continuous derivatives up to second order in the study area and the tensor of dyads type [3]:

$$\nabla^{*}\mathbf{a} = \mathbf{i}\nabla a_x + \mathbf{j}\nabla a_y + \mathbf{k}\nabla a_z$$

is the dual tensor

$$\nabla\mathbf{a} = \mathbf{i}\frac{\partial \mathbf{a}}{\partial x} + \mathbf{j}\frac{\partial \mathbf{a}}{\partial y} + \mathbf{k}\frac{\partial \mathbf{a}}{\partial z},$$

and I - identity tensor

$$\mathbf{I} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk},$$

For vector-valued functions a, b $\in W^2_2(\tau)$, and $\nabla\varphi \in L_2(\tau)$, $S \equiv \partial\tau$; $L \equiv \partial S$ due to the fundamental equations (16-21), we have the vector-tensor generalization of the integral theorems of analysis, such as: generalized Stoke's theorem

$$\iint_{(S)}\left(\mathbf{n}, \left[\nabla, \left[\mathbf{I}, \mathbf{a}\right]\right]\right)dS = \oint_{(L)}\left[d\mathbf{r}, \mathbf{a}\right], \quad (22)$$

since, from (20) we have

$$\iint_{(S)}\left(\mathbf{n}, \left[\nabla, \left[\mathbf{I}, \mathbf{a}\right]\right]\right)dS = \iint_{(S)}\left(\mathbf{n}, \left(\nabla^{*}\mathbf{a} - \mathbf{I}\left(\nabla, \mathbf{a}\right)\right)\right)$$

$$dS = \oint_{(L)}\left[d\mathbf{r}, \left[\mathbf{I}, \mathbf{a}\right]\right] = \oint_{(L)}\left[d\mathbf{r}, \mathbf{a}\right]$$

and the generalized theorem of Ostrogradsky - Gauss

$$\iiint_{(\tau)}\left\{\left(\nabla\left(\nabla, \mathbf{a}\right), \mathbf{b}\right) - \left(\mathbf{a}, \nabla\left(\nabla, \mathbf{b}\right)\right)\right\}d\tau = \quad (23)$$

$$\iint_{(S)}\left(\left\{\frac{\partial \mathbf{a}}{\partial \mathrm{n}} + \left[\mathbf{n}, \left[\nabla, \mathbf{a}\right]\right] - \mathbf{n}\left(\nabla, \mathbf{a}\right)\right\}, \mathbf{b}\right)dS =$$

$$= \oint_{(L)}\left(d\mathbf{r}, \left[\mathbf{a}, \mathbf{b}\right]\right) + \iint_{(S)}\left(\mathbf{a}, \left\{\frac{\partial \mathbf{b}}{\partial \mathrm{n}} + \left[\mathbf{n}, \left[\nabla, \mathbf{b}\right]\right] - \mathbf{n}\left(\nabla, \mathbf{b}\right)\right\}\right)dS$$

Then for the operators in (9-10) constructed generalized Green's formula:

$$\iiint_{(\tau)}\left\{\left(\nabla\left(\nabla, \mathbf{a}\right), \mathbf{\Gamma}\left(\left|\mathbf{x} - \mathbf{y}\right|\right)\right) - \left(\mathbf{a}, \nabla\left(\nabla, \mathbf{\Gamma}\left(\left|\mathbf{x} - \mathbf{y}\right|\right)\right)\right)\right\}d\tau =$$

$$= \iint_{(S)}\left\{\left(\left(\left(\frac{\partial \mathbf{a}}{\partial n} + \left[\mathbf{n}, \left[\nabla, \mathbf{a}\right]\right]\right), \mathbf{\Gamma}\right) - \left[\mathbf{n}, \mathbf{\Gamma}\right]\left(\nabla, \mathbf{a}\right)\right) - \left(\mathbf{a}, \left(\frac{\partial \mathbf{\Gamma}}{\partial n} + \left[\mathbf{n}, \left[\nabla, \mathbf{\Gamma}\right]\right]\right)\right)\right\}dS. \quad (24)$$

$$\mathbf{a} = \oiint_{(S+\Sigma)}\left\{\left(\left(\frac{\partial \mathbf{a}}{\partial n} + \left[\mathbf{n}, \left[\nabla, \mathbf{a}\right]\right]\right), \mathbf{\Gamma}\right) - \left(\mathbf{a}, \left(\frac{\partial \mathbf{\Gamma}}{\partial n} + \left[\mathbf{n}, \left[\nabla, \mathbf{\Gamma}\right]\right]\right)\right)\right\}dS. \quad (25)$$

$$\mathbf{V} = \oiint_{(S+\Sigma)}\left\{\left(\left(\frac{\partial \mathbf{V}}{\partial n} + \left[\mathbf{n}, \mathbf{\Omega}\right]\right), \mathbf{\Gamma}\right) - \left(\mathbf{V}, \left(\frac{\partial \mathbf{\Gamma}}{\partial n} + \left[\mathbf{n}, \left[\nabla, \mathbf{\Gamma}\right]\right]\right)\right)\right\}dS; \quad (26)$$

$$\mathbf{\Omega} = \oiint_{(S+\Sigma)}\left\{\left(\left(\frac{\partial \mathbf{\Omega}}{\partial n} + \left[\mathbf{n}, \left[\nabla, \mathbf{\Omega}\right]\right]\right), \mathbf{\Gamma}\right) - \left(\mathbf{\Omega}, \left(\frac{\partial \mathbf{\Gamma}}{\partial n} + \left[\mathbf{n}, \left[\nabla, \mathbf{\Gamma}\right]\right]\right)\right)\right\}dS; \quad (27)$$

$$\mathbf{\Psi} = \oiint\limits_{(S+\Sigma)} \left\{ \left(\left(\frac{\partial^* \mathbf{\Psi}}{\partial n}, \mathbf{\Gamma} \right) - [\mathbf{n}, \mathbf{\Gamma}](\nabla, \mathbf{\Psi}) \right) - \left(\mathbf{\Psi}, \left(\frac{\partial \mathbf{\Gamma}}{\partial n} + [\mathbf{n}, [\nabla, \mathbf{\Gamma}]] \right) \right) \right\} dS; \quad (28)$$

$$\mathbf{\Phi} = \oiint\limits_{(S+\Sigma)} \left\{ \left(\left(\frac{\partial^* \mathbf{\Phi}}{\partial n}, \mathbf{\Gamma} \right) - [\mathbf{n}, \mathbf{\Gamma}](\nabla, \mathbf{\Phi}) \right) - \left(\mathbf{\Phi}, \left(\frac{\partial \mathbf{\Gamma}}{\partial n} + [\mathbf{n}, [\nabla, \mathbf{\Gamma}]] \right) \right) \right\} dS / \quad (29)$$

Integrating in space (τ) a combination of differential operators (9, 7, 10, 13, 15) (for an arbitrary vector a), taking the standard limit, and also taking into account the properties of the double-layer potential [5], the fundamental solution of Laplace's equation in R^3, we have an integral representation solutions (25).

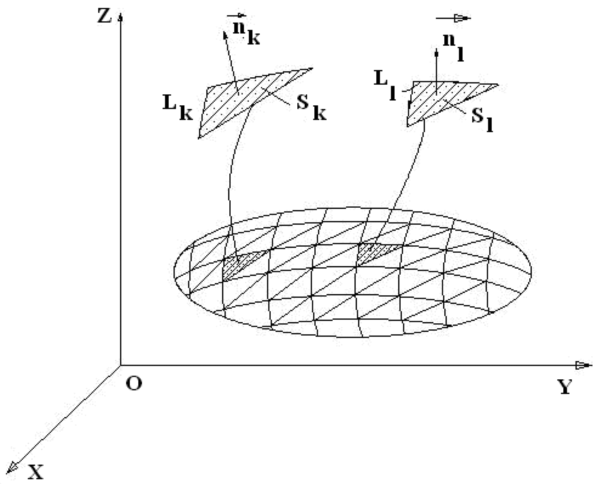

Fig.2 Elements of a triangulated surface (S).

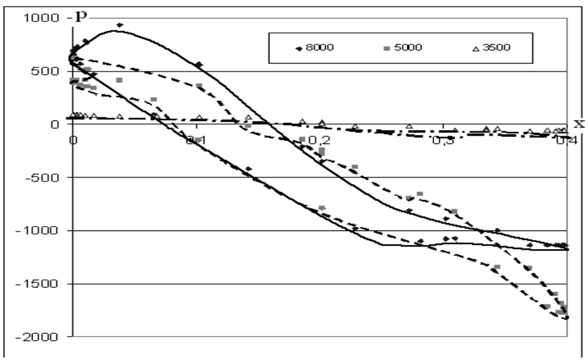

Fig.3 Influence of the number of cells of the triangulation surface of the ellipsoid (1.0, 1.0, 0.3) on the quality of the calculation at α = 5°.

This expression is the source of the integral representations of solutions of the definition of basic kinematics and dynamic characteristics of the interaction of a moving viscous fluid and a solid. And here all the differential operations are linear combinations of the characteristics of the problem and potentials Ψ and Φ: (26), (27), (28), (29).

Numerical solution of linear systems of boundary integral equations type (26 - 29) for kinematics and dynamic characteristics can be obtained using the classical quadrature integration on each element of the triangulated surface (Fig.2).

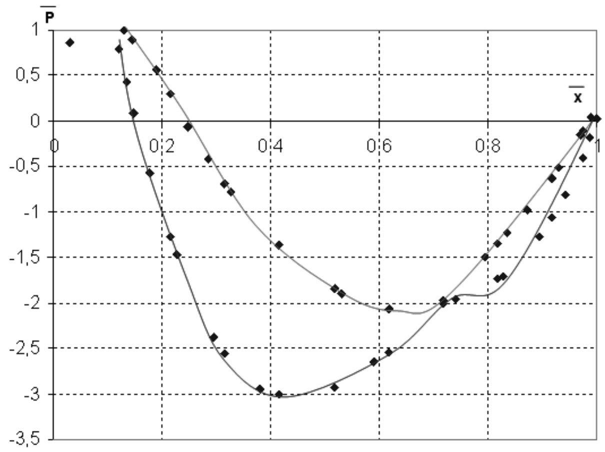

Fig.4 The pressure distribution on the top and bottom limits of longitudinal section of an ellipsoid $\overline{p} = f(\overline{x})$ with α = 10°.

Developed a process of triangulation of the ideology of the family of surfaces of second order and, in particular, the ellipsoidal shape, the algorithm and developed the appropriate software package, which is the basis of computer technology research of real fluidic characteristics of aircraft, vehicles and fortifications.

An analytical ideology of creating a local coordinate system on the elements of a triangulated surface of the three-dimensional body, and identified all the geometric characteristics of the elements of the triangulation.

We study the limiting values of improper integrals of the potentials of single and double layers, singular integrals and integrals of regular vector-valued functions, allowing to develop a range of quadrature formulas for the approximate calculation of these integrals on the elements of a triangulated surface of the three-dimensional body.

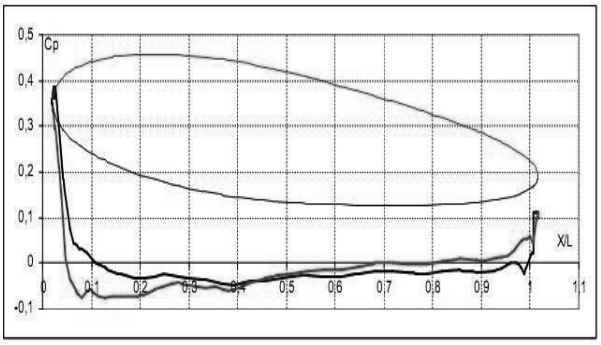

Fig.5 Distribution of pressure on the surface of an ellipsoid with elongation $\lambda = 5{,}33$ at $Re = 10^4$ and angle of attack $\alpha = 5°$.

On this basis, developed the algorithm and software packages numerical implementation of boundary integral equation method for solving boundary value problems of flow around the body of the spatial flow of a viscous incompressible fluid based on the full Navier-Stokes equations.

Some results of calculations for the distribution and total hydrodynamic characteristics, as well as the convergence of the computational method are shown in Fig. 3-7 [6].

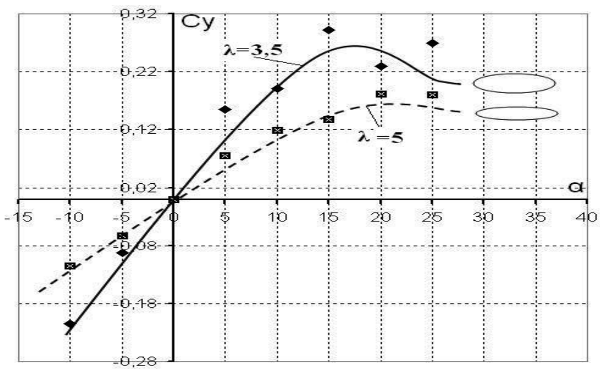

Fig.6 The dependence of the lift coefficient of an ellipsoid with different elongation.

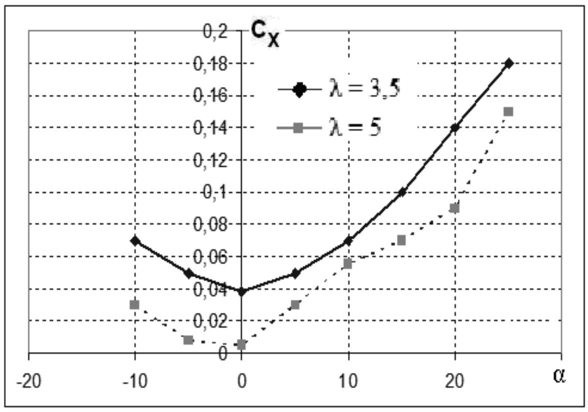

Fig.7 The dependence of the drag coefficient on the angle of attack ellipsoid at different extensions and $Re = 4.9 \cdot 10^4$.

RESULTS AND DISCUSSIONS

This approach has the absolute advantage over finite difference methods and finite element method. That is why at the present time this method has been successfully applied to solve complex engineering problems - flat and spatial, stationary and time dependent. Shown in Fig. 3-5 results illustrate the wide range of capabilities of the developed method in terms of computing as distributed and total hydrodynamic characteristics of flow around bodies of arbitrary space configuration real stream media including theology and any external influences.

CONCLUSIONS

In this paper, a new generalized approach to obtain analytical and numerical solutions for a wide class of nonlinear problems in continuum mechanics is presented. New approach and required formal apparatus for the formulation of the boundary integral equations, equivalent to the initial-boundary problems of the basic mathematical models in mechanics of fluid and gas, are developed.

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